Paying Tribute to James Eells and Joseph H. Sampson:

In Commemoration of the Fiftieth Anniversary of Their Pioneering Work on Harmonic Maps

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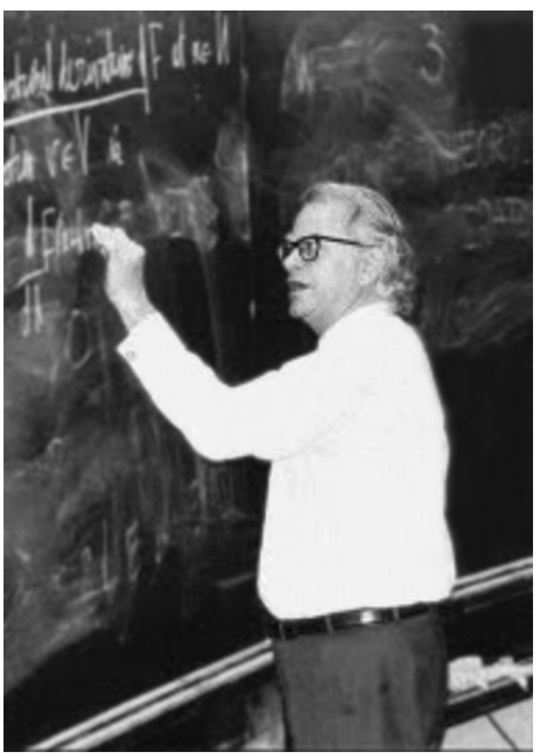

Professor J. Eells (1926–2007).

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James Eells was born in Cleveland, Ohio, in 1926 and passed away in Cambridge, England, in February 2007. He earned his PhD from Harvard University under the guidance of the topologist and analyst Hassler Whitney in 1954. He worked at the Institute for Advanced Study in Princeton and at the University of California in Berkeley from 1956–1958. He then returned to the East Coast where he accepted a position at Columbia University (1958–1964). He also taught at Churchill College, Cambridge, and Cornell University. Later, attracted by the atmosphere and potential at the University of Warwick, he joined the mathematics department there and became a professor of global analysis and geometry in 1969. Eells organized year-long highly successful Warwick symposia, namely, "Global Analysis" (1971–1972) and "Geometry of the Laplace Operator" (1976–1977). He was the first director of the Mathematical Section of the International Centre for Theoretical Physics in Trieste from 1986 to 1992. As a prominent professor and an inventive mathematician, Eells's mathematical influence in the field of harmonic

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maps became internationally recognized and widespread. According to the Mathematics Genealogy Project, he advised thirty-eight PhDs, who, in turn, produced 198 descendants. One of Eells's PhDs students was Andrea Ratto, who describes his experience as follows:

"When I first met Jim at Warwick in 1985, he gave me a copy of his survey article [9] (with L. Lemaire). He told me, 'We look for a candidate for the best map in a given homotopy class.' This is a fascinating problem, because it combines calculus of variations, differential equations, Riemannian geometry and topology. A special instance concerns the case of mappings between spheres, and Jim was intrigued by this topic, because in this context solutions are unstable and more difficult to obtain and thus less likely to be understandable. A few weeks after our first meeting, Jim told me that, in his opinion, the right topic for my research activity could be the development of the beautiful existence theory of R. T. Smith for equivariant harmonic maps of spheres. That was the beginning of my adventure in mathematics, and all that sprang from our conversations always remains, in my mind, tightly tied to the extraordinary charisma and exquisite mathematical taste of J. Eells."

James Eells retired from the University of Warwick in 1992. Then he moved to Cambridge and continued to work on harmonic maps for the remainder of his life.

Joseph H. Sampson was born in Philadelphia in 1926 and passed away in the South of France in July 2003. He earned his PhD from Princeton University under Salomon Bochner in 1951 (we often utilize Bochner's techniques in harmonic maps). Sampson then worked as a Moore Instructor at MIT. He was appointed as visiting assistant professor at the Johns Hopkins University in 1955 and then as assistant professor and was promoted to associate professor in 1963. He finally became full professor in 1965. He was an editor of the *American Journal of Mathematics* from 1978 to 1992 and the chair of the mathematics department at Johns Hopkins from 1969 to 1979. Sampson retired from Johns Hopkins in 1990. He was Yuan-Jen Chiang's advisor at Johns Hopkins, where she earned her PhD in 1989. For insights into her interactions with Sampson, Chiang provides the following:

"In 1982, when I first met Joe, I had only a Bachelor of Science degree. The requirements for a PhD included passing three oral examinations (each with two professors). For my three exams I chose real analysis, differential geometry, and algebraic topology. When I took Differential Geometry, Joe and I had arguments, which caused me to literally cry. Later, however, I recalled the questions and then realized that he was exceptionally brilliant. In 1984, and even to this day, I remember asking Joe the question, 'What is the goal and objective of your life?' Joe replied, 'Life is hostile, not evil. Finance is very important, but mathematics is most important to me.' Then I asked, 'Would you like to be my PhD advisor?' Joe then asked, 'Do you really want to study harmonic maps?' 'Yes!' I replied. Then we started a long journey which became an ordeal and Joe made me upset several times. He was bombastic. At first he wanted me to generalize (1) value distribution theory (VDT) for holomorphic maps of Riemann surfaces into harmonic maps and (2) VDT for harmonic maps of complex manifolds. Unfortunately, after approximately two years studying S. S. Chern's articles and investigating other related papers, I realized that such generalizations would not work. Essentially I was able to show Joe that such a generalization had an obstruction since the pull-back f^* commutes with ∂ and $\bar{\partial}$ for a holomorphic map, but not for a harmonic map. I remember that his face turned gray suddenly, and he said, 'Mathematics is difficult.' Then we investigated another topic, which did not work well either. Finally, he suggested 'Harmonic maps of V-manifolds' for me, and we both experienced a happy ending. I was so fortunate to be Joe's only PhD student to study harmonic maps with him directly (his other student, E.-B. Lin, went to MIT to study geometric quantization and came back for defense). I was very pleased to learn the beautiful tensor techniques from Joe through Eisenhart's book. I admired his penetrating insight and impeccable taste which characterized the precious guidance that he provided over those years."

Professor J. H. Sampson (1926–2003).

For more than four decades, Eells and Sampson were very good friends; both were outstanding mathematicians, brilliant teachers, and great experts in analysis, geometry, and topology. Sampson invited Eells to speak on harmonic maps at a mathematics department colloquium at Johns Hopkins around 1985. The joint paper of Chiang and Ratto [5], published in the *Bulletin of the French Mathematical Society* in 1992, was dedicated to Eells and Sampson. On July 20, 1992, Chiang received a letter from Eells: *Dear Dr. Chiang, I have just received my copy of Bull. SMF 120 and was most surprised and pleased to read a* *dedication by you and our excellent friend Andrea. What a nice idea! Thank you very much, and all my best. Yours cordially, James Eells.* That was the style of Jim: genuine, simple, and direct.

The theory of harmonic maps between Riemannian manifolds was first established by Eells and Sampson [14]. However, the notion of harmonic map was introduced by Sampson in the hope of obtaining a homotopy version of the highly successful Hodge theory for cohomology in 1952. Not long after that, his then colleague John Nash (one of the three Nobel laureates in economics in 1994) proposed a quite different but equivalent definition—both of them were Moore Instructors at MIT at that time. Fuller [15] also came upon harmonic maps in 1954. Later, when both were working at the Institute for Advanced Study at Princeton, Eells and Sampson wrote their famous and outstanding paper "Harmonic mappings of Riemannian manifolds" [14]. This article is considered as the pioneering work in the theory of harmonic maps and has provided the seeds for many further developments. As Eells pointed out in the preface of [7], "*Harmonic maps pervade differential geometry and mathematical physics: they include geodesics, minimal surfaces, harmonic functions, abelian integrals, Riemannian fibrations with minimal fibres, holomorphic maps between Kähler manifolds, chiral models, and strings.*"

A harmonic map $f : (M^m, g) \to (N^n, h)$ from an m-dimensional Riemannian manifold into an n-dimensional Riemannian manifold is defined as a critical point of the energy functional

$$E(f) = \frac{1}{2} \int_M |df|^2 dv,$$

where dv is the volume form of M determined by the metric g. The Euler-Lagrange equation associated with the energy functional is

$$\tau(f) = trace_g(\nabla df) = 0 \,, \tag{1}$$

where ∇df is the second fundamental form of the map and the left member of (1) is called the *tension field* of f. Equation (1) is a second-order semilinear system of partial differential equations: these analytical features can be deduced by the expression of (1) in local charts, i.e.,

$$g^{ij}(f^{\alpha}_{ij} - {}^{M}\Gamma^{k}_{ij} f^{\alpha}_{k} + {}^{N}\Gamma^{\alpha}_{\beta\gamma} f^{\beta}_{i} f^{\gamma}_{j}) = 0,$$

where the sum over repeated indices is understood and ${}^{M}\Gamma^{k}_{ij}$ and ${}^{N}\Gamma^{\alpha}_{\beta\gamma}$ denote the Christoffel symbols of the Levi-Civita connections on M and N, respectively.

In simple terms, the main idea of Eells and Sampson can be described as follows. Suppose that f_0 wraps (in a given homotopy class) a rubber (M) into a marble (N). If we deform f_0 (the initial condition) by using heat, then its energy decreases and we expect that the system evolves towards a configuration of minimum energy (i.e., a harmonic map). Note that this is the first example of a significant application of a *nonlinear* heat flow, where the nonlinearity has a strong geometrical meaning because it depends on the curvature of N. The novelty of this approach is that its success is strictly related to the curvature itself (and so, to the topology) of the target manifold N. In this sense, it is important to say that Eells-Sampson's Theorem was the first instance where curvature played a basic role in analysis, a fact which was a crucial milestone for global analysis. More specifically, Eells and Sampson [14] were able to prove the following assertion: *Let N have nonpositive Riemannian curvature and let $f : M \to N$ be a continuously differentiable map. Let f_t be the solution of the heat equation associated to* (1)*; i.e., $\tau(f_t) = \frac{\partial f}{\partial t}$, which reduces to f at $t = 0$. If f_t is bounded as $t \to \infty$, then f is homotopic to a harmonic map f' for which $E(f') \le E(f)$. In particular, if N is compact, then every continuous map $M \to N$ is homotopic to a harmonic map.*

In order to study geometric and topological problems, Hamilton introduced the Ricci flow with the aim of attacking the Thurston geometrization conjecture and the Poincaré conjecture. At the 2006 International Congress of Mathematicians in Madrid, he said that his initial inspiration came in the late 1960s, when he attended the seminars of Eells and Sampson on harmonic maps, who first suggested that one might be able to utilize evolution equations to study the Poincaré conjecture. Later, Hamilton used the idea of evolution equations and developed the theory of the Ricci flow, which laid the foundation for G. Perelman's work for solving both the Poincaré and the Thurston conjectures.

Shortly after the publication of [14], the explosion of interest in harmonic maps started (unless otherwise indicated, detailed bibliographical references for all the works that we mention can be found in [1], [9], and [10]: Hartman showed that *if $f : M \to N$ is harmonic with M compact and $Riem^N \le 0$ at every point of $f(M)$ and there is a point of $f(M)$ at which $Riem^N < 0$, then f is unique in its homotopy class.* The previous results of Eells and Sampson were extended to harmonic maps between manifolds with boundary by Hamilton. The curvature condition $Riem^N \le 0$ was modified by Sacks and Uhlenbeck as follows: *if $m = 2$ and $\pi_2(N) = 0$, then given a map $f_0 : M \to N$ there is a harmonic map f homotopic to f_0*. The condition $Riem^N \le 0$ can also be replaced by the condition that the image of f_0 (and hence of f) supports a uniformly strictly convex function (by Jost). Moreover, Schoen and Uhlenbeck proved a partial regularity theorem which asserts that a bounded energy minimizing map $f : M \to N$ between two Riemannian manifolds is regular (in the interior) except for a closed singular set S of Hausdorff dimension at most $(m-3)$. In the meantime, Giaquinta and Giusti and also Hildebrandt-Kaul-Widman proved a partial regularity theorem in the case that $f(M)$ is contained in a single chart of N. In the 1970s and 1980s, utilizing the ideas and methods of complex and algebraic geometries, Eells and Wood, Chern and Wolfson, Burstall and Wood, Eells and Salamon, etc., constructed and characterized harmonic maps from Riemann surfaces into projective spaces or complex Grassmannians. All these important developments were made easy to access, thanks to the precious surveys of Eells and Lemaire ([9], [10], and [11]). In the 1980s, Siu and Sampson [19], [21], and [22] made breakthroughs on harmonic maps of Kähler manifolds. Carlson and Toledo also studied harmonic maps of Kähler manifolds into locally symmetric spaces. In 1989, Uhlenbeck explored harmonic maps into Lie groups and obtained elegant results. Afterwards, two books on harmonic maps, integrable systems, conservation laws, and moving frames were published by Hélein [17], etc. In summary, a multitude of mathematicians, with profoundly different backgrounds, turned their research interest to harmonic maps.

Harmonic maps are intimately connected with the classical study of minimal and constant mean curvature submanifolds: that is not simply because a Riemannian immersion is minimal (i.e., a critical point of the volume functional) if and only if it is harmonic, but also because of the common methods in analysis. In this spirit, Eells and Ratto collaborated on a book [12] (published by Princeton University Press) about harmonic maps and minimal immersions such that the presence of suitable symmetries (due to invariance with respect to suitable group actions or isoparametric functions) reduces the analytical problem to the study of a nonlinear ODE system. Writing this monograph was an amazing experience for Ratto since he could really enjoy a deep and daily interaction with Jim. In particular, he could share Jim's view about the value of simple, key starting ideas and examples and, in a sort of ideally complementary vision, about the fact that all branches of mathematics (and the sciences) are just one thing. Jim could imagine, in the future, mathematical methods providing increasing support for all the other scientific disciplines, and that was a strong motivation for his research and for the enthusiastic guidance with which he encouraged and motivated his great number of students and collaborators. In that period, we also had the opportunity to work together on some extensions of the equivariant theory of spheres to the case of Euclidean ellipsoids [13]: Jim made me proud by telling me that he was extremely happy with our construction of a significant family of harmonic morphisms from suitable 3-dimensional ellipsoids to S^2. He considered these among the geometrically more interesting examples within the theory of harmonic morphisms, a field which now, thanks to the work of world-class mathematicians such as Wood and Baird (both former PhD students of Jim at Warwick), has attracted the interest of a very ample new generation of mathematicians (see [2] and references therein).

The notion of a harmonic map interestingly extends to the cases that manifolds are not smooth, but singular spaces. In this direction, Chiang studied "Harmonic maps of V-manifolds" [4] in 1990. The main difficulties arise from the complicated behavior near the singular locus of V-manifolds. Therefore, a new method is required to study spectral geometry of V-manifolds by applying the techniques in Baily's paper and Sampson's book manuscripts [20]. Then we use triangulations and simplices to deal with harmonic maps of V-manifolds. Afterwards, Chiang and Ratto [5] studied harmonic maps of spaces with conical singularities from a different approach in 1992 since their structures are different from those of V-manifolds. In the meantime, Gromov and Schoen [16] also investigated harmonic maps into singular spaces and p-adic superrigidity for lattices into groups of rank one. All these works motivated Eells's final monograph, *Harmonic Maps between Riemannian Polyhedra* [8], coauthored with Fuglede, on harmonic theory with singular domains and targets, which was published in 2001.

Wave maps are harmonic maps on Minkowski spaces and have been studied since the late 1980s. In this quarter, there have been many new developments achieved by a number of well-known mathematicians. Yang-Mills fields (cf. Donaldson [6]) are the critical points of the Yang-Mills functionals of connections whose curvature tensors are harmonic. They were first explored by a few physicists in the 1950s, and since then there have been many striking developments in this topic. Chiang's recent book *Developments of harmonic maps, wave maps and Yang-Mills fields into biharmonic maps, biwave maps and bi-Yang-Mills fields* [3] was published by Birkhäuser-Springer in 2013. Due to her feelings and personal experience, it has been an honor to conceive this book in the memory of Professors Eells and Sampson as follows: *The names of these two pioneers of the theory of harmonic maps will be engraved in the minds of all mathematicians who work on harmonic maps, wave maps and Yang-Mills fields, for their great and everlasting contributions.* Chiang met 2010 Fields Medalist C. Villani (director of the Institut Henri Poincaré in Paris) at the Second Pacific Rim Mathematics

Association Congress in Shanghai, China, in June 2013. He delivered a great and very impressive opening address, and he was one of six advisory editorial board members of her book. The citation for Villani's Fields Medal was "For his proofs of nonlinear Landau damping and convergence to equilibrium for the Boltzmann equation." Chiang also presented "Some properties of biwave Maps" at ICM 2010 in Hyderabad, India.

In the 1990s (see [3] for detailed references), Klainerman and Machedon and Klainerman and Selberg investigated the general Cauchy problem for wave maps, in any dimension greater than or equal to two, and obtained the almost optimal local well-posedness for regular data. In the difficult case of dimension two, Christodoulou and Tahvildar-Zadeh studied the regularity of spherically symmetric wave maps by imposing a convexity condition for the target manifold. Shatah and Tahvildar-Zadeh also studied the optimal regularity of equivariant wave maps into 2-dimensional rotationally symmetric and geodesically convex Riemannian manifolds. The study of the general wave maps problem incorporated methods that exploited the null-form structure of the wave map system, as in the work of Grillakis, as well as the geometric structure of the equations as done by Struwe. Keel and Tao studied the 1-(spatial) dimensional case. Tataru, following Tao, has used new techniques which allow one to treat the Cauchy problem with critical data. Their methods rely on harmonic analysis, such as adapted frequency and gauge-theoretic geometric techniques. Tao established the global regularity for wave maps from $\mathbf{R}\times\mathbf{R}^m$ into the sphere S^n for high (I) and low (II) dimensions m. Similar results were obtained by Klainerman and Rodnianski for target manifolds that admit a bounded parallelizable structure. Nahmod, Stefanov, and Uhlenbeck studied the Cauchy problem for wave maps from $\mathbf{R}\times\mathbf{R}^m$ into a (compact) Lie group (or Riemannian symmetric spaces) when $m \geq 4$ and established global existence and uniqueness, provided the Cauchy initial data are small in the critical norm. Shatah and Struwe obtained similar results simultaneously, also in the case that the target is any complete Riemannian manifold with bounded curvature. Recently, Kenig, Merle, and Duyckaerts investigated global well-posedness, scattering, and finite time blow-up. Furthermore, it is incredible that Tao wrote a book review [23] in the *Bulletin of the AMS* in October 2013, and he listed his five preprints about global regularity of wave maps III, IV, V, VI, and VII.

Other variant topics of harmonic maps are the objects of significant present research: for instance, biharmonic maps were first studied by Jiang [18] in 1986. Also, biwave maps (i.e., biharmonic maps on Minkowski spaces), bi-Yang-Mills fields, exponentially harmonic maps, exponential wave maps, and exponential Yang-Mills connections are topics of growing interests, (see [3]). Teichmüller theory, minimal surfaces, loop groups, integrable systems, etc., are among them as well.

Professors Eells and Sampson are gone, and the exemplars are in the past. As Jim pointed out that all branches of mathematics (and the sciences) are just one thing, he could have imagined that mathematical methods would provide increasing support for all the other scientific disciplines in the future. To commemorate the fiftieth anniversary of the Eells-Sampson pioneering work in harmonic maps in 2014, we sincerely hope that our generation of mathematicians will light the torch and carry on the Eells-Sampson legacy and spirit to the next generation of young mathematicians and scientists. It is not easy to predict now which developments, among those that we cited in this article, will be more fruitful in the future. But, for sure, we can say that the study of nonlinear evolution equations will continue to be a fundamental tool for the understanding of the central open problems in geometric analysis, partial differential equations, topology, and mathematical physics.

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